

Dynamics of Distal Actions on Certain Compact Spaces

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Abstract

We show that for a linear transformation T , on a Euclidean space, with determinant one, the following are equivalent: (1) T generates a compact group in the special linear group, (2) the corresponding action of T on the unit sphere is distal, and (3) the corresponding action on the real projective space is distal. For ‘affine’ actions on \mathbb{S}^1 , for different rotations, we discuss the existence and the behaviour of fixed points, whether they are attracting or repelling. We will then discuss dynamics of specific ‘affine’ actions on the n -sphere \mathbb{S}^n , $n > 1$, and on the p -adic unit sphere \mathcal{S}_n , $n \in \mathbb{N}$, for ‘affine’ actions.

Keywords: dynamical system; distal action; affine map; fixed point

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1 Introduction

Let X be a (Hausdorff) topological space. Let $T : X \rightarrow X$ be a homeomorphism. The map T is said to be distal if for any two distinct points $x, y \in X$, the closure of double orbit $\{(T^n(x), T^n(y)) \mid x, y \in X, n \in \mathbb{Z}\}$ in $X \times X$ stays away from the diagonal, i.e. for $x, y \in X, x \neq y$, $\overline{\{(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\}} \cap \{(d, d) \mid d \in X\} = \emptyset$. The notion of distality was introduced by Hilbert (cf. Ellis [4], Moore [8]) and studied by many in different context (see Abels [1]-[2], Furstenberg [5], Raja-Shah [9]-[10] and Shah [11], and references cited therein).

Ellis characterized distal maps on compact spaces. If X is a compact Hausdorff space and T is a surjective continuous map then T is distal if and only if $E(T)$, the closure of $\{T^n \mid n \in \mathbb{N}\}$ in X^X is a group where X^X endowed with weak topology (cf. Ellis [4], Moore [8]). If T is distal then T is injective and hence a homeomorphism. Furstenberg has obtained a structure theorem for distal flows on compact (metrizable) spaces. For a

compact space, T is distal if and only if $T^n, n \in \mathbb{N}$ is distal. If G is a locally compact (Hausdorff) topological group with the identity e and $T \in \text{Aut}(G)$, then T is distal if and only if $e \notin \overline{\{T^n(x)\}_{n \in \mathbb{Z}}}, \forall x \neq e$. The contraction group for $T \in \text{Aut}(G)$, is defined as $C(T) = \{x \in G \mid T^n(x) \rightarrow e\}$. Note that if T is distal then the contraction groups $C(T)$ and $C(T^{-1})$ are trivial. The converse is shown to hold on all locally compact groups recently (more generally, see Raja-Shah [10]).

Here we characterise the class of distal transformations on real projective spaces \mathbb{RP}^n corresponding to linear maps. A real projective space \mathbb{RP}^n is the set of lines passing through the origin in \mathbb{R}^{n+1} , the Euclidean space of dimension $n+1$. A relation $x \sim y$ if $x = \alpha y$ for $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$ and $\alpha \in \mathbb{R}^*$, is an equivalence relation. For $x \in \mathbb{R}^{n+1} \setminus \{0\}$, let $[x]$ denote the equivalence class of x in \mathbb{RP}^n , where $[x] = \{y \in \mathbb{R}^{n+1} \setminus \{0\} : x \sim y\}$. For any $T \in GL(n+1, \mathbb{R})$, let $\tilde{T} : \mathbb{RP}^n \rightarrow \mathbb{RP}^n$ be such that $\tilde{T}([x]) = [T(x)]$, for $[x] \in \mathbb{RP}^n$. It is well defined since T is linear. Note that $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$, endowed with the quotient topology, is a compact space and it is homeomorphic to \mathbb{S}^n / \approx , where \approx identifies antipodal points on the n -sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ here the norm $\|\cdot\|$ is the usual norm on \mathbb{R}^{n+1} .

We now define some maps which will be useful. For $T \in GL(n+1, \mathbb{R})$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$, let $\overline{T} : \mathbb{S}^n \rightarrow \mathbb{S}^n$ be defined as $\overline{T}(\overline{x}) = T(x)/\|T(x)\|$, where $\overline{x} = x/\|x\| \in \mathbb{S}^n$. Here, \overline{T} is well defined as T is linear. Let $\rho : \mathbb{S}^n \rightarrow \mathbb{RP}^n$ is defined as $\rho(\overline{x}) = [x]$. Observe that both \overline{T} and \tilde{T} are homeomorphisms.

In section 2 we shall prove that for any $T \in SL(n+1, \mathbb{R})$, the distality of T on \mathbb{S}^n , the distality of \tilde{T} on \mathbb{RP}^n and the condition that T generates a compact group are equivalent (see Theorem 2.1). In section 3 we discuss the ‘affine’ action of \overline{T}_a on \mathbb{S}^n defined by $\overline{T}_a(x) = (a + T(x))/\|a + T(x)\|$, $x \in \mathbb{S}^n$. Here, \overline{T}_a is well defined and a homeomorphism if $T \in GL(n+1, \mathbb{R})$ keeps \mathbb{S}^n invariant and $a \in \mathbb{R}^{n+1}$ with $\|a\| < 1$ (see Lemma 3.1). We show that for a rotation map T by an element belonging to a specific region in \mathbb{S}^1 , which depends on a , \overline{T}_a on \mathbb{S}^1 has fixed points and study the behaviour, whether the fixed points are attracting or repelling. We will then study, for $T = Id$ (resp. $T = -Id$), the similar behaviour of fixed points of ‘affine’ action \overline{T}_a (resp. \overline{T}_a^2) on \mathbb{S}^n , $n > 1$ (see Corollary 3.4).

For the p -adic field \mathbb{Q}_p we show that if $T \in GL(n, \mathbb{Q}_p)$, which keeps the p -adic unit sphere $\mathcal{S}_n \subset \mathbb{Q}_p^n$ invariant, if $\|a\|_p < 1$ (resp. $\|a\|_p > 1$) the corresponding ‘affine’ action on \mathcal{S}_n is distal (resp. has a unique fixed point which is attracting), see subsection 3 for definition and more details. More generally the behaviour of such ‘affine’ action on \mathcal{S}_n is different from that on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

2 Dynamics of the maps \overline{T} and \tilde{T}

In this section we consider the corresponding actions of a linear map $T \in SL(n+1, \mathbb{R})$ on \mathbb{S}^n and \mathbb{RP}^n . Note that if $T \in SL(n+1, \mathbb{R})$ is distal, it does not imply that \overline{T} and \tilde{T} are distal. For example if we consider $T = (a_{ij})$, a 2×2 matrix such that $a_{11} = 1, a_{12} = 1, a_{21} = 0$

and $a_{22} = 1$. Then it is easy to check that T is distal on \mathbb{R}^2 but both \overline{T} on \mathbb{S}^1 and \tilde{T} on \mathbb{RP}^1 are not distal. However if T keeps \mathbb{S}^n invariant, then T generates a compact group in $SL(n+1, \mathbb{R})$ and its corresponding actions on $\mathbb{R}^{n+1}, \mathbb{S}^n$ and \mathbb{RP}^n are distal. The following theorem shows that the converse for the action of T on \mathbb{RP}^n essentially holds. Note that $SL(n, \mathbb{R})$ is closed in $M(n, \mathbb{R})$, the set of all $n \times n$ matrices equipped with the usual norm as it is isomorphic to \mathbb{R}^{n^2} . Since $M(n, \mathbb{R})$ is finite dimensional, all vector space norms on it are equivalent; $SL(n+1, \mathbb{R})$ is a topological group w.r.t. this topology.

Theorem 2.1. *Let $T \in SL(n+1, \mathbb{R})$. Let $\overline{T} : \mathbb{S}^n \rightarrow \mathbb{S}^n$ and $\tilde{T} : \mathbb{RP}^n \rightarrow \mathbb{RP}^n$ be the corresponding maps. Then the following are equivalent:*

- (a) \overline{T} is distal.
- (b) \tilde{T} is distal.
- (c) T generates a compact group.

Proof. (a) \Rightarrow (b) : Let \overline{T} be distal. If possible, suppose that \tilde{T} is not distal. Then there exist $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$, $[x] \neq [y]$ and an unbounded sequence $\{n_k\} \subset \mathbb{N}$ such that $\tilde{T}^{n_k}([x]) \rightarrow [a]$ and $\tilde{T}^{n_k}([y]) \rightarrow [a]$, for some $a \in \mathbb{R}^{n+1} \setminus \{0\}$. This implies that $[T^{n_k}(x)] \rightarrow [a]$ and $[T^{n_k}(y)] \rightarrow [a]$ or $\rho(\overline{T}^{n_k}(\overline{x})) \rightarrow [a]$ and $\rho(\overline{T}^{n_k}(\overline{y})) \rightarrow [a]$ (where $\rho : \mathbb{S}^n \rightarrow \mathbb{RP}^n$ is defined as $\rho(\overline{x}) = [x]$ as defined in the introduction).

As \mathbb{S}^n is compact, there exists a subsequence of $\{n_k\}$, which we denote by $\{n_k\}$ again, such that $\overline{T}^{n_k}(\overline{x}) \rightarrow \overline{b}$ and $\overline{T}^{n_k}(\overline{y}) \rightarrow \overline{c}$ in \mathbb{S}^n , for some $b, c \in \mathbb{R}^{n+1} \setminus \{0\}$. This implies that $\rho(\overline{b}) = [a]$ and $\rho(\overline{c}) = [a]$. This implies that $\overline{b} = \pm \overline{c}$. Hence, as \overline{T} is distal, we have $\overline{x} = \overline{y}$ or $\overline{x} = -\overline{y}$. This implies that $[x] = [y]$, which is a contradiction. Hence \tilde{T} is distal.

(b) \Rightarrow (c) : Let \tilde{T} be distal. We show that T generates a compact group. As $\det(T) = 1$, then at least one of the following holds: (i) all the eigen values of T are of absolute value one, (ii) at least one eigen value of T has absolute value less than one and at least one eigen value of T has absolute value greater than one.

If possible, suppose $\{T^n\}_{n \in \mathbb{N}}$ is not relatively compact in $SL(n+1, \mathbb{R})$. Then there exists a subsequence $\{m_k\} \subset \mathbb{N}$ such that $\{T^{m_k}\}$ is divergent, i.e. it has no convergent subsequence. Moreover, we show that there exists a subsequence of $\{m_k\}$, which we denote by $\{m_k\}$ again, and a non zero vector v_0 such that $\{T^{m_k}(v_0)\}$ converges.

Suppose (i) holds. If 1 is an eigen value of T then there exists a non-zero eigen vector v such that $T(v) = v$. If all the eigen values of T are complex and of absolute value one, then there exists a two dimensional subspace W' such that $T|_{W'}$ is a rotation map and generates a compact group in $GL(W')$. Hence if (i) holds there always exists a non zero vector v_0 .

Now suppose (ii) holds. Then T has at least one eigen value of absolute value less than one, and hence the contraction group $C(T)$ of T is non trivial, and we can choose v_0 as any non zero vector in $C(T)$.

By Lemma 2.1 of Dani-Raja [3], there exist a subspace W of \mathbb{R}^{n+1} and a subsequence $\{n_k\}$ of $\{m_k\}$ such that $\{T^{n_k}(v)\}$ converges, for every $v \in W$ and $T^{n_k}(v) \rightarrow \infty$ (i.e.

$\|T^{n_k}(v)\| \rightarrow \infty$), whenever $v \notin W$. Here $W \neq \{0\}$ as $v_0 \in W$.

Now we show that $W = \mathbb{R}^{n+1}$. If possible suppose $W \neq \mathbb{R}^{n+1}$, then there exists $u \in \mathbb{R}^{n+1} \setminus W$ such that $T^{n_k}(u) \rightarrow \infty$. Since \mathbb{S}^n is compact, passing to a subsequence if necessary, we have $T^{n_k}(u)/\|T^{n_k}(u)\| \rightarrow a$, for some $a \in \mathbb{S}^n$. Let $v_0 \in W$ be as above. As $u \notin W, u + v_0 \notin W$ and therefore $T^{n_k}(u + v_0) \rightarrow \infty$. As $\{T^{n_k}(v_0)\}$ is bounded and $T^{n_k}(u + v_0) \rightarrow \infty$, we have $T^{n_k}(v_0)/\|T^{n_k}(u + v_0)\| \rightarrow 0$ and $\|T^{n_k}(u)\|/\|T^{n_k}(u) + T^{n_k}(v_0)\| \rightarrow 1$. This implies that $T^{n_k}(u + v_0)/\|T^{n_k}(u + v_0)\| \rightarrow a$. This shows that $\overline{T^{n_k}(u)} \rightarrow a$ and $\overline{T^{n_k}(u + v_0)} \rightarrow a$. Here, $[u] \neq [u + v_0]$ as $v_0 \in W$ and $u \notin W$. Therefore $\tilde{T}^{n_k}([u]) \rightarrow [a]$ and $\tilde{T}^{n_k}([u + v_0]) \rightarrow [a]$. This is a contradiction as \tilde{T} is distal. Hence $W = \mathbb{R}^{n+1}$. This implies that $\{T^{n_k}\}$ is bounded. As $\{n_k\}$ is a subsequence of $\{m_k\}$, we arrive at a contradiction to our earlier assumption that $\{T^{m_k}\}$ is divergent. Hence $\{T^n\}_{n \in \mathbb{N}}$ is relatively compact and all its limit points belong to $SL(n+1, \mathbb{R})$ as $\det(T) = 1$. Therefore, $H = \overline{\{T^n\}_{n \in \mathbb{N}}}$ is a compact semigroup in $SL(n+1, \mathbb{R})$ and hence H is a compact group (see Sec. 1 in Ch. A of Hofmann-Mostert [6]).

It is easy to see that (c) \Rightarrow (a), as compact groups act distally. \square

Remark: (i) From Theorem 2.1, it follows that for $T \in SL(n+1, \mathbb{R})$ the distality of \overline{T} or \tilde{T} implies the distality of T on \mathbb{R}^{n+1} .

(ii) If $T \in GL(n+1, \mathbb{R})$, there exists $\alpha \in \mathbb{R}$ such that $\alpha T \in SL(n+1, \mathbb{R})$ and as $[\alpha T] = [T]$ and $\overline{\alpha T} = \overline{T}$, the action of both T and αT on \mathbb{S}^n , as well as, on \mathbb{RP}^n are same.

3 Dynamics of ‘affine’ maps on \mathbb{S}^n

In this section, we first consider the ‘affine’ action of \overline{T}_a on \mathbb{S}^n , which is defined as $\overline{T}_a(x) = a + T(x)/\|a + T(x)\|, x \in \mathbb{S}^n$, where $T \in GL(n+1, \mathbb{R})$, which keeps \mathbb{S}^n invariant and $a \in \mathbb{R}^{n+1}$ with $0 < \|a\| < 1$. We study the dynamics of \overline{T}_a on \mathbb{S}^n , for $T(x) = \pm Id$. It is easy to show that a non trivial homeomorphism on \mathbb{S}^1 with a fixed point is not distal. We will talk about the periodic points of \overline{T}_a on \mathbb{S}^1 for different rotation and reflection maps T . Note that, the set of T in $GL(n, \mathbb{R})$ which keeps \mathbb{S}^n invariant is the group of orthogonal matrices $O(2, \mathbb{R})$, and therefore T is either a rotation or a reflection. Here $\{\overline{T}_a^n\}_{n \in \mathbb{N}}$ is orientation preserving if T is a rotation but $\{\overline{T}_a^n\}_{n \in \mathbb{N}}$ is orientation reversing if T is a reflection.

Lemma 3.1. *For $a \in \mathbb{R}^{n+1}$ with $\|a\| < 1$ and $T \in GL(n+1, \mathbb{R})$ which keeps \mathbb{S}^n invariant, the map \overline{T}_a on \mathbb{S}^n is a homeomorphism.*

Proof. From the definition, it is clear that \overline{T}_a is continuous. It is enough to show that \overline{T}_a is bijection since any continuous bijection on compact Hausdorff space is a homeomorphism. Suppose $x, y \in \mathbb{S}^n$ such that $\overline{T}_a(x) = \overline{T}_a(y)$. Then we have $(a + T(x))/\|a + T(x)\| = (a + T(y))/\|a + T(y)\|$ or $(1 - \beta)a = \beta T(y) - T(x)$, where $\beta = \|a + T(x)\|/\|a + T(y)\|$. This implies $\|a\| \geq 1$ unless $T(x) = T(y)$. Hence \overline{T}_a is one-one.

Let $y \in \mathbb{S}^n$ fixed. Let $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $\Psi(t) = \|tT^{-1}(y) - T^{-1}(a)\|$. Here, Ψ is a continuous map, and hence the image of Ψ is connected. Note that as T keeps \mathbb{S}^n invariant, T is an isometry. We have $\Psi(0) = \|T^{-1}(a)\| = \|a\| < 1$ and $\Psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Therefore there exists a $t_0 \in \mathbb{R}^+$, $t_0 \neq 0$ such that $\Psi(t_0) = 1$. Let $x = t_0 T^{-1}(y) - T^{-1}(a)$. Then $x \in \mathbb{S}^n$ and $\bar{T}_a(x) = y$. Hence \bar{T}_a is surjective. \square

Note that, \mathbb{R}^2 is isomorphic to the field \mathbb{C} of complex numbers and \mathbb{S}^1 is a group under multiplication. For $x \in \mathbb{R}^2 \setminus \{0\}$, we take x^{-1} as inverse of x in \mathbb{C} .

Proposition 3.2. *If $T(x) = rx$ is a rotation map on S^1 and $a \in \mathbb{R}^2 \setminus \{0\}$ with $\|a\| < 1$, then $\bar{T}_a(x) = (a + rx)/\|a + rx\|$ admits a fixed point if and only if $r_1 \geq \sqrt{1 - \alpha^2}$ and $|r_2| \leq \alpha$, where $r = r_1 + ir_2$ and $\alpha = \|a\|$. Moreover, if fixed points of \bar{T}_a , for $r = (r_1, r_2)$ and $a = (a_1, a_2)$, exist then they are of the form-*

$$a(t - r)^{-1}, \text{ where } t = r_1 \pm \sqrt{r_1^2 - (1 - \alpha^2)}.$$

Proof. Suppose \bar{T}_a has a fixed point, say, x_0 . Then $(a + rx_0)/\|a + rx_0\| = x_0$ or $a + rx_0 = bx_0$, where $b = \|a + rx_0\|$. Since $\|x_0\| = 1$ and for $r = r_1 + ir_2$, we get $a = [(b - r_1) - ir_2]x_0$. Therefore b satisfies a quadratic equation $b^2 - 2br_1 + 1 - \alpha^2 = 0$. As $\alpha < 1$ and $b > 0$, we have that $r_1 > 0$. From the above equation we get that $b = r_1 \pm \sqrt{r_1^2 - (1 - \alpha^2)}$. As $r_1, b \in \mathbb{R}^+$, we have that $r_1 \geq \sqrt{1 - \alpha^2}$ and $|r_2| \leq \alpha$. Conversely, suppose $r_1 \geq \sqrt{1 - \alpha^2}$ and $|r_2| \leq \alpha$. As $r_1 \geq \sqrt{1 - \alpha^2} > 0$, $t = r_1 \pm \sqrt{r_1^2 - (1 - \alpha^2)}$ are positive real numbers, for which $\|t - r\| = \alpha$. If we choose $x_t = a(t - r)^{-1}$, then $\|x_t\| = 1$ and $\bar{T}_a(x_t) = x_t$. Note that, \bar{T}_a has only one fixed point if $r = (\sqrt{1 - \alpha^2}, \pm\alpha)$. \square

Let X be a topological space. Let p be a fixed point of a continuous function $f : X \rightarrow X$. The point p is said to be an *attracting* (resp. *repelling*) fixed point if there exists an open set U containing p such that if $x \in U$, then $f^n(x) \rightarrow p$ as $n \rightarrow \infty$ (resp. if $x \in U$ and $x \neq p$, then there exists a positive integer n such that $f^n(x) \notin U$).

Theorem 3.3. *Let T , a , α , \bar{T}_a as in Proposition 3.2, for $r_1 = \sqrt{1 - \alpha^2}$, \bar{T}_a has only one fixed point which is an attracting fixed point, and for $r_1 > \sqrt{1 - \alpha^2}$, \bar{T}_a has two fixed points; one of them is attracting and other is repelling.*

Proof. Note that the fixed points of \bar{T}_a , for the rotation r , are $x_t = a(t - r)^{-1}$, where $t = r_1 \pm \sqrt{r_1^2 - (1 - \alpha^2)}$.

Observe that, $\bar{T}_{sa}(sx) = s\bar{T}_a(x)$, for $s \in \mathbb{S}^1$. Hence x_0 is a fixed point of \bar{T}_a if and only if sx_0 is a fixed point of \bar{T}_{sa} . Moreover, $\bar{T}_a^n(x) \rightarrow x_0$ if and only if $\bar{T}_{sa}^n(sx) \rightarrow sx_0$. In particular, x_0 is an attracting (resp. repelling) fixed point of \bar{T}_a if and only if sx_0 is an attracting (resp. repelling) fixed point of \bar{T}_{sa} . Therefore without loss of generality, we can replace a by sa , where $s = \bar{a}^{-1}(0, 1)$, for $(0, 1) \in \mathbb{S}^1$ and assume that $a = (0, \alpha)$. We may also assume that the fixed points of \bar{T}_a are $x_{t_1} = (-r_2/\alpha, \sqrt{r_1^2 - (1 - \alpha^2)}/\alpha)$ and $x_{t_2} = (-r_2/\alpha, -\sqrt{r_1^2 - (1 - \alpha^2)}/\alpha)$.

Step I. Let $\psi : [0, 1] \rightarrow \mathbb{S}^1$ defined by $\psi(t) = e^{2\pi i(\theta+t)}$, where $e^{2\pi i\theta} = x_{t_1}$. Here, the restriction of ψ to $]0, 1[$ is a homeomorphism and $\psi(0) = \psi(1) = x_{t_1}$. Then there exists $s_0 \in]0, 1[$ such that $\psi(s_0) = x_{t_2}$. Let $\phi : [0, 1] \rightarrow [0, 1]$ be a homeomorphism defined by $\phi(t) = (\psi^{-1} \circ \overline{T}_a \circ \psi)(t)$, for $0 < t < 1$ and $\phi(0) = 0$, $\phi(1) = 1$. Observe that $\psi \circ \phi = \overline{T}_a \circ \psi$ and the set of fixed points of ϕ is $\{0, s_0, 1\}$.

Let $r_1 = \sqrt{1 - \alpha^2}$. Then $x_{t_1} = x_{t_2}$ and \overline{T}_a has only one fixed point. For any $t \in]0, 1[$, we have either $\phi(t) < t$ or $\phi(t) > t$. Moreover, as ϕ is increasing, $\{\phi^n(t)\}_{n \in \mathbb{N}}$ is a decreasing or an increasing sequence which converge to fixed point 0 or 1. Therefore, for all $x \in \mathbb{S}^1$, $\overline{T}_a^n(x) \rightarrow x_{t_1} = x_{t_2}$ which is an attracting fixed point for \overline{T}_a .

Step II. Now consider $r_1 > \sqrt{1 - \alpha^2}$ then $|r_2| < \alpha$. Let $-\alpha < r_2 < 0$.

As $\sqrt{r_1^2 - (1 - \alpha^2)} < r_1$, there exist $s_2, s_3 \in]0, s_0[$ such that $\psi(s_2) = -r^{-1}\overline{a}$, $\psi(s_3) = -\overline{a}$. Then as $\overline{T}_a(-r^{-1}\overline{a}) = -\overline{a}$, we have that $\phi(s_2) = s_3 < s_2$. Since there are no other fixed points between 0 and s_0 , $\phi(t) < t$, for all $t \in]0, s_0[$. As ϕ is an increasing function, $\{\phi^n(t)\}_{n \in \mathbb{N}}$ is a decreasing sequence, for all $t \in]0, s_0[$. That is, $\{\phi^n(t)\}_{n \in \mathbb{N}}$ converges to a fixed point and therefore $\phi^n(t) \rightarrow 0$, for all $t \in]0, s_0[$.

Let $s_1 \in]0, 1[$ such that $\psi(s_1) = (1, 0)$, where $(1, 0) \in \mathbb{S}^1$. As $\overline{T}_a((1, 0)) = ((0, \alpha) + (r_1, r_2)) / \|(0, \alpha) + (r_1, r_2)\| = (r_1, r_2 + \alpha) / \sqrt{1 + \alpha^2 + 2r_2\alpha}$. This implies that $\phi(s_1) \in]s_0, 1[$ and $\phi(s_1) > s_1$. Since ϕ has no fixed point in $]s_0, 1[$, $\phi(t) > t$ for all $t \in]s_0, 1[$ and hence $\phi^n(t) \rightarrow 1$, for all $t \in]s_0, 1[$.

Therefore $\overline{T}_a^n(x) \rightarrow x_{t_1}$, for all $x \in \mathbb{S}^1 \setminus \{x_{t_2}\}$; i.e. x_{t_1} is an attracting fixed point and x_{t_2} is a repelling fixed point for \overline{T}_a .

Step III. Let $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is defined by $\varphi((x_1, x_2)) = (-x_1, x_2)$, and $\overline{T}'_a(x) = (a + r'x) / \|a + r'x\|$, where $r' = r_1 - ir_2$ and $x \in \mathbb{S}^1$. It is easy to see that $\varphi \circ \overline{T}_a = \overline{T}'_a \circ \varphi$, as φ is a linear map. Therefore the dynamics of \overline{T}_a and \overline{T}'_a will be same, i.e. $\overline{T}_a^n(x) \rightarrow y$ if and only if $\overline{T}'_a^n(\varphi(x)) \rightarrow \varphi(y)$. Hence for $0 < r_2 < \alpha$, x_{t_1} is an attracting and x_{t_2} is a repelling fixed point for \overline{T}_a .

□

For non trivial T , such affine actions on higher dimensional torus is complex. Here we discuss the dynamics of \overline{T}_a on $\mathbb{S}^n, n > 1$, for $T = \pm Id$.

Corollary 3.4. Suppose $T \in GL(n+1, \mathbb{R})$ which keeps \mathbb{S}^n invariant. Let $a \in \mathbb{R}^{n+1} \setminus \{0\}$ with $\|a\| < 1$. Let $\overline{T}_a : \mathbb{S}^n \rightarrow \mathbb{S}^n$ be defined as $\overline{T}_a(x) = (a + T(x)) / \|a + T(x)\|, x \in \mathbb{S}^n$.

(i) For $T(x) = x$, \overline{a} and $-\overline{a}$ are the only fixed points of \overline{T}_a and $\overline{T}_a^m(x) \rightarrow \overline{a}, \forall x \in \mathbb{S}^n \setminus \{-\overline{a}\}$ i.e. \overline{a} is an attracting fixed point and $-\overline{a}$ is a repelling fixed point. In particular \overline{T}_a is not distal.

(ii) For $T(x) = -x$, $\overline{a}, -\overline{a}, x_0$ and $a - x_0$ are the only periodic points of \overline{T}_a of period two where x_0 is such that $\|a - x_0\| = 1$ and $\{\overline{T}_a^{2m}(x)\}$ converges to either x_0 or $a - x_0$, for all $x \in \mathbb{S}^n \setminus \{\overline{a}, -\overline{a}\}$. In other words, $x_0, a - x_0$ are attracting fixed points and $\overline{a}, -\overline{a}$ are repelling fixed points. In particular \overline{T}_a is not distal.

Proof. Step 1. Let $T(x) = x$. By Theorem 3.3, it is easy to see that \bar{a} is an attracting fixed point and $-\bar{a}$ is a repelling fixed point for \bar{T}_a . That is, $\bar{T}_a^n(x) \rightarrow \bar{a}$, for all $x \in \mathbb{S}^1 \setminus \{-\bar{a}\}$.

Step 2. Let $n \in \mathbb{N}$. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on \mathbb{R}^{n+1} . Consider W_x , a vector space generated by a and a fixed $x \in \mathbb{S}^n$ such that $x \neq \delta a$, for any non zero real number δ . It is clear that W_x is \bar{T}_a -invariant. Here, W_x is a two dimensional vector space. There exists a nonzero vector $y \in W_x$ with $\|y\| = 1$ such that $\langle x, y \rangle = 0$ and $\{x, y\}$ forms a basis of W_x . For any vector $z = (z_1, z_2, \dots, z_{n+1}) \in W_x$, we can find scalars α_1 and α_2 such that $z = \alpha_1 x + \alpha_2 y$ and $\|z\|^2 = \sum z_i^2 = \langle \alpha_1 x + \alpha_2 y, \alpha_1 x + \alpha_2 y \rangle = \alpha_1^2 + \alpha_2^2$. Let $\Phi : W_x \rightarrow \mathbb{R}^2$ be defined as $\Phi(z) = (\alpha_1, \alpha_2)$, $z \in W_x$. Then Φ is an isometry and in particular $\Phi(\mathbb{S}^n \cap W_x) = \mathbb{S}^1$, a unit circle in \mathbb{R}^2 . From Step 1, $\bar{T}_a^m(x) \rightarrow \bar{a}$. Since this holds for any $x \in \mathbb{S}^n \setminus \{-\bar{a}\}$, the assertion in (i) holds.

Step 3. Let $T(x) = -x$. Then $\bar{T}_a(x) = (a - x) / \|a - x\|$. The set of periodic points of \bar{T}_a of order two is $\{\bar{a}, -\bar{a}, x_0, a - x_0\}$, where x_0 is as in the hypothesis.

Consider, $n = 1$. Let $x \in \mathbb{S}^1 \setminus \{\bar{a}, -\bar{a}\}$ be fixed. Let ϑ (resp. ϑ') be an angle between a and x (resp. $-a$ and x), and ϑ_m (resp. ϑ'_m) be an angle between a and $\bar{T}_a^m(x)$ (resp. $-a$ and $\bar{T}_a^m(x)$), $m \in \mathbb{N}$. Then $\cos \vartheta = \langle a, x \rangle / \|a\| \|x\| = \langle a, x \rangle / \alpha$.

$$\cos \vartheta_1 = \langle a, (a - x) / \|a - x\| \rangle / \alpha = (\alpha - \cos \vartheta) / c_1, \text{ where } c_1 = \|a - x\|.$$

$$\cos \vartheta_2 = \langle a, \bar{T}_a((a - x) / c_1) \rangle / \alpha = ((c_1 - 1) \alpha + \cos \vartheta) / c_2, \text{ where } c_2 = \|(c_1 - 1)a + x\|.$$

$$\text{As } c_2 \geq 1 - |c_1 - 1| \alpha, \cos \vartheta_2 - |c_1 - 1| \alpha \cos \vartheta_2 \leq (c_1 - 1) \alpha + \cos \vartheta.$$

If $c_1 < 1$, then we get that $\cos \vartheta_2 < \cos \vartheta$. That is,

$$\text{if } \|a - x\| < 1 \text{ then } \|\bar{a} - \bar{T}_a^2(x)\| > \|\bar{a} - x\|. \quad (1)$$

Similarly, we can show that, if $c_1 > 1$, $\cos \vartheta_2' < \cos \vartheta'$, where $\cos \vartheta' = \langle -a, x \rangle / \alpha$ and $\cos \vartheta_2' = \langle -a, \bar{T}_a^2(x) \rangle / \alpha$. That is,

$$\text{if } \|a - x\| > 1 \text{ then } \|\bar{a} + \bar{T}_a^2(x)\| > \|\bar{a} + x\|. \quad (2)$$

Let $\psi : [0, 1] \rightarrow \mathbb{S}^1$ defined by $\psi(t) = e^{2\pi i(r+t)}$, where $e^{2\pi i r} = \bar{a}$. Here, the restriction of ψ to $]0, 1[$ is a homeomorphism and $\psi(0) = \psi(1) = \bar{a}$. Then there exist $t_0, t_1 \in]0, 1[$ such that $\psi(t_0) = x_0$ and $\psi(t_1) = a - x_0$. Interchanging x_0 and $a - x_0$, if necessary. We may assume that $t_0 < t_1$. Note that as $\|a - x_0\| = 1$ and $\phi(1/2) = -\bar{a}$, we get that $t_0 < 1/2 < t_1$. Let $\phi : [0, 1] \rightarrow [0, 1]$ be a homeomorphism defined by $\phi(t) = (\psi^{-1} \circ \bar{T}_a^2 \circ \psi)(t)$, for $0 < t < 1$ and $\phi(0) = 0, \phi(1) = 1$. Observe that $\psi \circ \phi = \bar{T}_a^2 \circ \psi$ and the set of fixed points of ϕ is $\{0, t_0, t_1, 1\}$. From (1), it follows that if $t \in]0, t_0[$ (resp. $t \in]t_1, 1[$), then $\phi(t) > t$ (resp. $\phi(t) < t$). Since ϕ is monotone, and in particular, increasing, it follows that $\{\phi^m(t)\}_{m \in \mathbb{N}}$ is an increasing sequence (resp. a decreasing sequence) which is bounded above (resp. below) by t_0 (resp. t_1), if $t \in]0, t_0[$ (resp. $t \in]t_1, 1[$), and hence it converges to a fixed point in $]0, t_0[$ (resp. $]t_1, 1[$). That is, $\phi^m(t) \rightarrow t_0$ for $t \in]0, t_0[$ and $\phi^m(t) \rightarrow t_1$ for $t \in]t_1, 1[$. Similarly using (2) we get that, if $t \in]t_0, 1/2[$ (resp. $t \in]1/2, t_1[$), $\phi(t) < t$ (resp. $\phi(t) > t$).

Therefore, $\phi^m(t) \rightarrow t_0$ for $t \in]t_0, 1/2[$ and $\phi^m(t) \rightarrow t_1$ for $t \in]1/2, t_1[$. As ψ is continuous and $\psi \circ \phi = \overline{T}_a^2 \circ \psi$, the following holds: If $x \in \psi(]0, 1/2[)$ (resp. $x \in \psi(]1/2, 1[)$) then $\overline{T}_a^{2m}(x) \rightarrow x_0, \overline{T}_a^{2m+1}(x) \rightarrow a - x_0$ (resp. $\overline{T}_a^{2m}(x) \rightarrow a - x_0, \overline{T}_a^{2m+1}(x) \rightarrow x_0$). This shows that $x_0, a - x_0$ are attracting fixed points for \overline{T}_a^2 and $\overline{a}, -\overline{a}$ are repelling fixed points for \overline{T}_a^2 . In particular, \overline{T}_a is not distal. Hence (ii) holds for $n = 1$.

Using the above assertion for $n = 1$, and arguing as in step 2, we can show, this holds for all $n \in \mathbb{N}$. □

We know that for $r = (-1, 0)$ and $T(x) = rx$, \overline{T}_a has four periodic points of order two. The following Lemma shows that there exists a neighbourhood U of $(-1, 0)$ such that for every $r \in U$, \overline{T}_a has four periodic points of order two.

Lemma 3.5. *Let $T(x) = rx$, for $r = r_1 + ir_2$, on \mathbb{S}^1 , and $a, \alpha, \overline{T}_a$ as in Proposition 3.2. If $r_1 > 0$ and $|r_2| > \alpha$ then \overline{T}_a has no periodic point of order two. However, there exists a neighbourhood U of $(-1, 0)$ such that for $(r_1, r_2) \in U$, \overline{T}_a has four periodic points of order two.*

Proof. Here $\overline{T}_a(x) = (a + rx)/b_1$ and $\overline{T}_a^2(x) = (b_1a + ra + r^2x)/b_2$, where $b_1 = \|a + rx\|$ and $b_2 = \|b_1a + ra + r^2x\|$. Without loss of any generality we may assume that $a = (0, \alpha)$.

Step I. Consider $r_1 > 0$ and $r_2 > \alpha$. Let $\psi : [0, 1] \rightarrow \mathbb{S}^1$ be such that $\psi(t) = e^{2\pi i(1/4-t)}$ and $\psi(0) = \psi(1) = (0, 1)$. Let $\beta = \psi^{-1}(\overline{T}_a^{-1}(r^{-1}\overline{a}))$. Then for $x \in \psi([0, \beta[)$ $\psi^{-1}(x) < \psi^{-1}(\overline{T}_a^2(x))$. For $x = (x_1, x_2) \in \psi([\beta, 1])$ such that $x_2 \leq 0$, and $\overline{T}_a^2(x) = (y_1, y_2)$ such that $y_2 > 0$. This implies that \overline{T}_a has no periodic point of order two. In case of $r_1 > 0$ and $r_2 < -\alpha$, we can take $r' = r_1 - ir_2$ and \overline{T}'_a as in Step III of Theorem 3.3, and argue as above for r' and \overline{T}'_a and get that \overline{T}_a , and hence, \overline{T}_a has no periodic point of order two.

Step II. $\overline{T}_a^2(1, 0) = (x_1, x_2)$, where $x_1 = (r_1^2 - r_2^2 - r_2\alpha)/b_2, x_2 = (b_1\alpha + r_1\alpha + 2r_1r_2)/b_2$.

$\overline{T}_a^2(0, 1) = (y_1, y_2)$, where $y_1 = (-2r_1r_2 - r_2\alpha)/b_2, y_2 = (r_1^2 - r_2^2 + r_1\alpha + b_1\alpha)/b_2$.

$\overline{T}_a^2(0, -1) = (z_1, z_2)$, where $z_1 = (2r_1r_2 - r_2\alpha)/b_2, z_2 = (r_2^2 - r_1^2 + r_1\alpha + b_1\alpha)/b_2$.

As $\alpha < 1$ there exists a neighbourhood U of $(-1, 0)$ such that if $(r_1, r_2) \in U$ with $r_2 > 0, x_i > 0, y_i > 0$ and $z_i < 0$ for $i = 1, 2$. Let $U = \{(x, y) \mid x \leq -\sqrt{1 - \epsilon^2}, |y| < \epsilon\}$ be a neighbourhood of $(-1, 0)$ such that $\epsilon = \min\{\lambda_1, \lambda_2, \mu_1, \mu_2, \nu\}$, where

$$\begin{aligned}\lambda_1 &= 1/(\alpha + 2) \\ \lambda_2 &= (\alpha\sqrt{1 + \alpha^2} - \alpha)/2 \\ \mu_1 &= \sqrt{1 - \alpha^2}/4 \\ \mu_2 &= \sqrt{(1 - \alpha^2)/2} \\ \nu &= \sqrt{(1 - \alpha^2)/(\alpha + 2)}.\end{aligned}$$

It is easy to check that $\epsilon = \lambda_2$. Let $E = \{(x_1, x_2) \in \mathbb{S}^1 \mid x_1 \geq 0, x_2 \geq 0\}$ and $F = \{(x_1, x_2) \in \mathbb{S}^1 \mid x_1 \geq 0, x_2 \leq 0\}$. Consider E^0 , the interior of the set E .

$$\overline{T}_a^2((1,0)) \in E^0 \text{ as } r_2 < \min\{\lambda_1, \lambda_2\}.$$

$$\overline{T}_a^2((0,1)) \in E^0 \text{ as } r_2 < \min\{\mu_1, \mu_2\}.$$

$$\overline{T}_a^2((0,-1)) \in (F \cup E)^c \text{ as } r_2 < \nu.$$

Therefore, as $E, F \subset \mathbb{S}^1$ are compact and connected, and \overline{T}_a^2 is injective, we get that $\overline{T}_a^2(E) \subsetneq E$ and $F \subsetneq \overline{T}_a^2(F)$. As both E and F are homeomorphic to a closed interval in \mathbb{R} , considering the continuous map on the closed interval corresponding to \overline{T}_a^2 , we get that \overline{T}_a^2 has fixed points in E , i.e. there exists a $x = (x_1, x_2) \in E$ such that $\overline{T}_a^2(x) = x$ and $\overline{T}_a^2(\overline{T}_a(x)) = \overline{T}_a(x)$. Similarly, there exists $y \in F$ such that y and $\overline{T}_a(y)$ are periodic points of \overline{T}_a of order two.

For the rotation $r = (r_1, r_2) \in U$ with $r_2 < 0$, we can take $r' = r_1 - ir_2$ and \overline{T}'_a as in Step III of proof of Theorem 3.3, and use the above argument for r' and \overline{T}'_a we can deduce that \overline{T}'_a , and hence \overline{T}_a has four periodic points of order two. \square

Lemma 3.6. *Let $T \in O(2, \mathbb{R})$ be a reflection and $a, \alpha, \overline{T}_a$ as in Proposition 3.2. Then \overline{T}_a has two fixed points. In particular, \overline{T}_a is not distal.*

Proof. It is easy to see that the reflections across any lines through the origin are conjugate i.e. for given two reflections T_1 and T_2 there exists a rotation such that $T_1 = R_\theta T_2 R_{\theta-1}$, where R_θ and $R_{\theta-1}$ are rotations by θ angle in positive and negative direction respectively. Let $s = R_\theta(1,0)$, then $(\overline{T}_1)_{sa}(sx) = s(\overline{T}_2)_a(x)$. Now as in the proof of Theorem 3.3, without loss of generality we may assume that T is the reflection across y -axis.

It is easy to check that if a is orthogonal to reflection line i.e. $T(a) = -a$, then $(\alpha/2, \pm\sqrt{1-\alpha^2/4})$ are the fixed points, and if a is on y -axis i.e. $T(a) = a$, then $\pm\bar{a}$ are the fixed points of \overline{T}_a . Now let a be such that $T(a) \neq \pm a$. Let K (resp. L) be the closed arc between \bar{a} and $T(\bar{a})$ (resp. $-\bar{a}$ and $T(-\bar{a})$) with the shorter arc length, then $\overline{T}_a(K) \subsetneq K$ and $L \subsetneq \overline{T}_a(L)$. Hence arguing as in Step-II of Lemma 3.5 above, we get that \overline{T}_a has a fixed point in K , as well as, in L . Therefore \overline{T}_a is not distal. \square

Remark: For any reflection T , as \overline{T}_a has (two) fixed points and \overline{T}_a^2 is orientation preserving, the rotation number of \overline{T}_a^2 can only be 1. Hence \overline{T}_a may have periodic points of order at most two. For example, if $T(a) = -a$, $\pm\bar{a}$ are the periodic points of order two. The behaviour of the map \overline{T}_a near fixed/periodic point is complex. Therefore we would not discuss it in this case.

3.1 Dynamics of ‘affine’ maps on \mathcal{S}_n :

Let \mathbb{Q}_p^n be a n -dimensional p -adic vector space equipped with the p -adic norm defined as $\|(x_1, \dots, x_n)\|_p = \max\{\|x_1\|_p, \dots, \|x_n\|_p\}$. We now consider the ‘affine’ action \overline{T}_a on

\mathcal{S}_n , and discuss the dynamics of \overline{T}_a , which is different from the real case, where $\mathcal{S}_n = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{Q}_p^n \mid \|x\|_p = 1\}$. Note that, for $n = 1$, $\mathcal{S}_1 = \mathbb{Z}_p^* = \{x \in \mathbb{Q}_p \mid \|x\|_p = 1\}$.

Theorem 3.7. *Suppose $T \in GL(n, \mathbb{Q}_p)$ such that T keeps \mathcal{S}_n invariant. Let $\overline{T}_a : \mathcal{S}_n \rightarrow \mathcal{S}_n$ be defined as $\overline{T}_a(x) = \|a + T(x)\|_p \cdot (a + T(x))$, $\|a\|_p \neq 1$, where $\|\cdot\|_p$ denotes the p -adic norm. Then the following statements hold:*

- (i) *If $\|a\|_p < 1$ then \overline{T}_a is distal.*
- (ii) *If $\|a\|_p > 1$, then there exists $c \in \mathcal{S}_n$, such that for all $x \in \mathcal{S}_n$, $\overline{T}_a^m(x) \rightarrow c$. In particular, \overline{T}_a has a unique fixed point which is an attracting fixed point and \overline{T}_a is not distal.*

Proof. It is easy to see that \overline{T}_a is a homeomorphism as \overline{T}_a is of the form $\overline{T}_a(x) = a + T(x)$, for $\|a\|_p < 1$ and $\overline{T}_a(x) = \|a\|_p(a + T(x))$, for $\|a\|_p > 1$.

Case I. Suppose that $\|a\|_p < 1$.

$\overline{T}_a(x) = a + T(x)$. Clearly, $\overline{T}_a^m(x) = a + T(a) + \dots + T^m(x) \in \mathcal{S}_n$ for all $n \geq 1$. Hence for $x, y \in \mathcal{S}_n$, $\|\overline{T}_a^m(x) - \overline{T}_a^m(y)\|_p = \|T^m(x) - T^m(y)\|_p = \|T^m(x - y)\|_p = \|x - y\|_p$ as T keeps \mathcal{S}_n invariant. This implies that \overline{T}_a is distal.

Case II. Suppose that $\|a\|_p > 1$. Let $\alpha = \|a\|_p = p^{m_0}$, for some natural number $m_0 > 1$.

$$\begin{aligned}
\overline{T}_a(x) &= \|a\|_p(a + T(x)) \\
\overline{T}_a(x) &= \alpha a + \alpha T(x) \\
\overline{T}_a^2(x) &= \alpha a + \alpha^2 T(a) + \alpha^2 T^2(x) \\
&\vdots \\
\overline{T}_a^m(x) &= \alpha a + \alpha^2 T(a) + \dots + \alpha^m T^m(x) \\
&= \left(\sum_{k=0}^{m-1} \alpha^{k+1} T^k(a) \right) + \alpha^m T^m(x), m \geq 1
\end{aligned}$$

Note that, as $\|\alpha\|_p < 1$ and $\|T^m(x)\|_p = 1$, $\{\alpha^m T^m(x)\}$ converges to 0. The series $\sum_{k=0}^{\infty} \alpha^{k+1} T^k(a)$ converges to c (say). Then $\overline{T}_a^m(x) \rightarrow c$. As c does not depend on the choice of x , c is the unique fixed point and it is an attracting fixed point of \overline{T}_a . In particular, \overline{T}_a is not distal. □

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